Asymptotic localization of stationary states in the nonlinear Schrödinger equation

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The mapping of the nonlinear Schrödinger equation with a random potential on the Fokker-Planck equation is used to calculate the localization length of its stationary states. The asymptotic growth rates of the moments of the wave function and its derivative for the linear Schrödinger equation in a random potential are computed analytically, and resummation is used to obtain the corresponding growth rate for the nonlinear Schrödinger equation and the localization length of the stationary states.

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I. INTRODUCTION

In this work we consider problem of one-dimensional Anderson localization [1,2] for the nonlinear Schrödinger equation (NLSE)[3–5]. In spite of extensive research, many fundamental problems are still open, and, in particular, it is not clear whether in one dimension (1D) Anderson localization can survive the effects of nonlinearities. This problem is relevant for experiments in nonlinear optics—for example, disordered photonic lattices [6,7], where Anderson localization was found in the presence of nonlinear effects as well as experiments on Bose-Einstein condensates (BECs) in disordered optical lattices [8–12]. The interplay between disorder and nonlinear effects leads to new interesting physics [9,10,13–16]. In particular, the problems of the spreading of wave packets and transmission are not simply related [17,18], in contrast with the linear case.

We consider one-dimensional localization of stationary solutions of the NLSE in a random δ-correlated potential $V(x)$ with a Gaussian distribution (white noise), of zero mean and variance $2D$: namely, $\langle V(x)V(x')\rangle = 2D\delta(x-x')$. The linear version of this model was studied extensively in the past [19]. Following our previous analysis [20] we study Anderson localization of stationary solutions with energies ω in the framework of the stationary NLSE

$$\omega\phi(x) = -\partial_x^2\phi(x) + \beta\phi^3(x) + V(x)\phi(x), \tag{1}$$

where $\phi(x)$ is chosen real, since the stationary states are localized. The variables are chosen in dimensionless units and the Planck constant is $\hbar = 1$. For the lattice version of the model, it was established rigorously [3,21,22] that the stationary solutions of this equation are exponentially localized for a wide range of conditions. It was also argued that the rate of growth of moments for the stationary NLSE (1) coincides exactly with the linear case [20] and determines the localization length.

We will specifically calculate $\langle\phi^2(x)\rangle$ of solutions of Eq. (1) that are found for a certain ω, with given boundary conditions at some point—for example, $\phi(x=0)$ and $\phi'(x=0)$, where the prime means the derivative with respect to x. This will be done with the help of the analogy with the Langevin equation [19,20,23]. In particular, we will calculate the growth rate of the second moment,

$$2\gamma = \lim_{x\to\infty} \frac{\ln\langle\phi^2(x)\rangle}{x} > 0, \quad \xi = \frac{1}{\gamma}, \tag{2}$$

which will turn out to be independent of β, where ξ is the localization length. Note that it is different from the usually studied (in the linear case) self-averaging quantity $\gamma_s = \frac{1}{2}\frac{d}{dx}\langle\ln\phi^2(x)\rangle = \frac{1}{2}\lim_{x\to\infty}\frac{\langle\ln\phi^2(x)\rangle}{x}$, and γ is a smooth function of energy. Since the distribution of the random potentials is translationally invariant, it is independent of the choice of the initial point as $x=0$. As in the linear case, starting from a specific initial condition, $\phi(x)$ will typically grow. For specific values of ω at some point this function will start to decay, so that a normalized eigenfunction is found. This is the approach in the manner of Borland [24,25], which was made rigorous for the linear case in [26,27]. Here, following [20] we extend this approach, in a heuristic form, to the nonlinear case. The envelope of the wave function will grow exponentially if we start either from the right or from the left. The value of ω results from the matching condition, so that an eigenfunction has some maximum and decays in both directions as required by the normalization condition. The exponential decay is an asymptotic property, while the matching is determined by the potential in the vicinity of the maximum. This observation is crucial for the validity of this approach and enables us to determine the exponential decay rate of states from the solution of the initial value problem (1). In [20] a linear equation for the moments of $\phi(x)$ of Eq. (1) and its Langevin analog was derived and it was shown that the exponents that control the growth of the moments are identical to the ones of the linear system ($\beta=0$). In the present paper it will be shown that this is also correct for the asymptotic behavior of the moments.

The outline of the paper is as follows. The analogy with the Langevin equation and dynamics of the moments are outlined in Sec. II. The generalized Lyapunov exponents, which are the eigenvalues that determine the growth of the various moments, are presented in Sec. II, and their asymptotic behavior is derived analytically in Appendix A (they were found numerically in [28]). The resulting asymptotic expansion for the growth of the moments is presented in Sec. III. The results are summarized in Sec. IV.

II. FOKKER-PLANCK EQUATION AND LYAPUNOV EXPONENTS

Following Ref. [20], we perform the calculation of $\langle \phi^2(x) \rangle$ by using the analogy with the classical Langevin equation [19,23]. Therefore, considering the x coordinate as the formal time variable on the half axis $x \equiv \tau \in [0, \infty)$, Eq. (1) reduces to the Langevin equation

$$\ddot{\phi} + \omega\phi - \beta\phi^3 - V(\tau)\phi = 0, \tag{3}$$

with the δ correlated Gaussian noise $V(\tau)$. Now we introduce new variables $u=\phi$ and $v=\dot{\phi} \equiv \frac{d\phi}{d\tau}$ (which play the role of position and velocity in the Langevin equation) and a distribution function of these new variables is $P=P(u,v,\tau)$. The dynamical process in the presence of the Gaussian δ-correlated noise is described by the distribution function that satisfies the Fokker-Planck equation (FPE)[20,30]:

$$\partial_\tau P - [\omega u - \beta u^3]\partial_v P + v\partial_u P - Du^2 \partial_v^2 P = 0, \tag{4}$$

where Du^2 is the only nonzero component of the diffusion tensor.

We are interested in the average quantum probability density $\langle \phi^2(x) \rangle \equiv \langle u^2(\tau) \rangle$, where

$$\langle u^2(\tau) \rangle = \int u^2 P(u,v,\tau) du\, dv.$$

From the FPE we obtain a system of equations for the moments:

$$M_{k,l} = \langle u^k v^l \rangle, \tag{5}$$

where $k,l=0,1,2,\ldots$. Multiplying $u^k v^l$ by the FPE and integrating over u and v, one obtains the following relation for $M_{k,l}$:

$$\dot{M}_{k,l} = -l\omega M_{k+1,l-1} + kM_{k-1,l+1} + l(l-1)DM_{k+2,l-2} + \beta l M_{k+3,l-1}, \tag{6}$$

where $M_{k,l}$ with negative indices are assumed to vanish. We note that only terms with the same parity of $k+l$ are coupled. Since we are interested in $M_{2,0}=\langle u^2 \rangle$, we study only the case when this parity is even: namely, $k+l=2n$ with $n=1,2,\ldots$. The sum of the indices of the moments is $2n$, except the last term $\beta l M_{k+3,l-1}$, where the sum is $2(n+1)$. This leads to the infinite system of linear equations that can be written in the form

$$\dot{\mathbf{M}} = \mathcal{W}\mathbf{M}, \tag{7}$$

where the column vector is $\mathbf{M} = (M_{2,0}, M_{1,1}, M_{0,2}, M_{4,0}, M_{3,1}, \ldots)$ and $\mathcal{W}$ is the corresponding matrix. The matrix elements $\mathcal{W}_{k,l}$ are determined by Eq. (6). The solutions of the system of linear equations (7) are linear combinations of the eigenfunctions at time τ:

$$\mathbf{M}_\lambda(\tau) = \exp(\lambda\tau)\mathbf{U}_\lambda, \tag{8}$$

where $\mathbf{U}_\lambda = \mathbf{M}_\lambda(t=0)$ is the eigenvector of $\mathcal{W}$ corresponding to the eigenvalue λ found from the equation

$$\mathcal{W}\mathbf{U}_m = \lambda_m \mathbf{U}_m, \quad \mathbf{U}_m \equiv \mathbf{U}_{\lambda_m}. \tag{9}$$

The infinite matrix $\mathcal{W}$ consists of two parts. The first one is independent of β and consists of independent diagonal blocks A_n of size $(2n+1)\times(2n+1)$. The second one consists of the β-dependent terms which couple the nth and $(n+1)$th blocks and are located above the $(n+1)$th block and to the right of the nth block. Consequently, the β-dependent terms do not affect the characteristic polynomial, as can be shown by elementary operations on determinants. Therefore, the characteristic polynomial of $\mathcal{W}$ reduces to a product of the block determinants [20]

$$\prod_{n=1}^{\infty} \det(A_n - \lambda I_n) = 0, \tag{10}$$

where I_n is an $(2n+1)\times(2n+1)$ unit matrix. The diagonal block A_n of the infinite matrix $\mathcal{W}$ defined in Eq. (7), which couples the moments of order $2n$ with one another, is a band-diagonal square matrix of size $2n+1$. The explicit form of this matrix is given by

$$A_n = \begin{pmatrix} 0 & 2n & 0 & & & & & & \\ -\omega & 0 & 2n-1 & 0 & & & & & \\ 2D & -2\omega & 0 & 2n-2 & 0 & & & & \\ 0 & 6D & -3\omega & 0 & 2n-3 & 0 & & & \\ & & \cdot & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & 0 & (2n-2)(2n-1)D & -(2n-1)\omega & 0 & 1 \\ & & & & & 0 & (2n-1)2nD & -2n\omega & 0 \end{pmatrix}. \tag{11}$$

Let us denote by $\lambda_{\max}(n)$ the maximal eigenvalue of this matrix. The vector of the moments in this block is $\mathbf{M}_n=(M_{2n,0},M_{2n-1,1},\ldots,M_{2n-l,l},\ldots,M_{0,2n})$. In Appendix A it is proven that for $\omega=0$ the maximal Lyapunov exponent $\lambda_{\max}(n)$ behaves for large n as

$$\lambda_{\max}(n) \simeq \frac{3}{4}D^{1/3}(2n)^{4/3}. \tag{12}$$

Then it is argued and verified numerically that also for other values $\omega \neq 0$ it behaves in this way.

III. ASYMPTOTIC GROWTH OF THE MOMENTS

A. Eigenvalue problem for the moments

Taking into account Eqs. (8) and (9), we present the solution of Eq. (7) as an expansion

$$\mathbf{M}(\tau)=\sum_m C_m(\tau)\mathbf{U}_m=\sum_m e^{\lambda_m\tau}c_m\mathbf{U}_m, \tag{13}$$

where $c_m \equiv C_m(\tau=0)$. Due to the block structure, the eigenvectors are characterized by two indices $m=(n,k)$, where n indicates the number of blocks, while $k=1,2,\ldots,2n+1$ counts elements inside each block. Therefore, the eigenstates $\mathbf{U}_m \equiv \mathbf{U}_{n,k}$ are found from the following algorithm. For the block $n=1$ there are three eigenvalues $\lambda_{1,k}$ with corresponding eigenvectors $\bar{\mathbf{U}}_{1,k}$ determined by the first block A_1. Therefore, $\mathbf{U}_{1,k}=(\bar{\mathbf{U}}_{1,k},\mathbf{O})$, where $\mathbf{O}$ is an infinite zero vector. For the second block $n=2$ there are five eigenvalues $\lambda_{2,k}$ with corresponding eigenvectors $\mathbf{U}_{2,k}=(\mathbf{R}_{1,k},\bar{\mathbf{U}}_{2,k},\mathbf{O})$, where $\mathbf{R}_{1,k}$ is a 3D vector, while $\bar{\mathbf{U}}_{2,k}$ is a 5D vector, and $k=1,2,\ldots,5$. Here $\lambda_{2,k}$ and $\bar{\mathbf{U}}_{2,k}$ are determined from the second diagonal block matrix A_2, while $\mathbf{R}_{1,k}$ is determined by A_1 and by the corresponding β-dependent off diagonal block. Continuing this procedure, we obtain $2n+1$ eigenvectors for $\lambda_{n,k}$ in the form

$$\mathbf{U}_{n,k}=(\mathbf{R}_{n-1,k},\bar{\mathbf{U}}_{n,k},\mathbf{O}), \tag{14}$$

where $\mathbf{R}_{n-1,k}$ is a (n^2-1)-dimensional vector determined by $n-1$ diagonal and off-diagonal blocks of the truncated matrix $\mathcal{W}$.

Summation over m in Eq. (13) is broken into the sum over the block numbers $n\in[1,\infty)$ and the sum over indexing inside each block $l\in[0,2n]$. Thus, Eq. (13) reads

$$\mathbf{M}(\tau)=\sum_{n=1}^{\infty}\sum_{l=0}^{2n}c_{n,l}e^{\lambda_{n,l}\tau}\mathbf{U}_{n,l}. \tag{15}$$

The vector $\mathbf{M}$ consists of the block vectors $\mathbf{M}_n$: $\mathbf{M}=(\mathbf{M}_1,\mathbf{M}_2,\ldots,\mathbf{M}_n,\ldots)$, where $\mathbf{M}_n$ is a vector of $2n+1$ elements defined in Eq. (5) and corresponds to the moments of the order of $2n$. Therefore, the initial vector at $\tau=0$ is

$$(\mathbf{M}_1,\mathbf{M}_2,\ldots,\mathbf{M}_n,\ldots)=\sum_{n'}\sum_{l'}c_{n',l'}\mathbf{U}_{n',l'}. \tag{16}$$

Assume that at the initial point $\tau=0$ the wave function and its derivative are small, of the order of ϵ in units of $\sqrt{|\omega/\beta|}$. Then, at that point the moments scale as $\mathbf{M}_n \sim \epsilon^{2n}$ with ϵ arbitrary small. As follows from Eq. (16), one finds $c_{n,l}U_{n,l}\sim \bar{c}_n\epsilon^{2n}+o(\epsilon^{2n})$ with bounded $\bar{c}_n$ for any n, as demonstrated in Appendix B. In the linear case ($\beta=0$) the growth rate of each moment of the order of $2n$ corresponding to the nth block is determined by the eigenvalue with the largest real part, and we denote it by $\lambda_{\max}(n)\equiv\max(\mathrm{Re}\,\lambda_{n,l})$, where the maximum is over the $2n+1$ eigenvalues corresponding to the nth block, indicated by l. As shown in Appendix A, the asymptotic behavior of the generalized Lyapunov exponent $\lambda_{\max}(n)$ when $n\to\infty$ is given by $\lambda_{\max}(n)\sim \mathcal{A}n^{4/3}$, where $\mathcal{A}=\frac{3}{4}2^{4/3}D^{1/3}$ is a constant [see Eq. (12)]. The leading contribution to the growth of $\mathbf{M}_n(\tau)$ in the nonlinear case ($\beta\neq 0$) is determined by the sum

$$\tilde{\mathbf{M}}_n(\tau)=\sum_{m\geqslant n}\bar{c}_m\epsilon^{2m}e^{\mathcal{A}m^{4/3}\tau}, \tag{17}$$

as is clear from Eqs. (13)–(16).

B. Resummation

This series in Eq. (16) has a vanishing radius of convergence and probably has to be interpreted as an asymptotic series. It can be used to study the behavior of $\tilde{\mathbf{M}}_n(\tau)$ after being resummed. Such a resummation is done with the help of the identity

$$\exp(K^2)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}du\,\exp(-u^2+2Ku), \tag{18}$$

known as the Hubbard-Stratonovich transformation. We can rewrite the above series as follows:

$$\begin{aligned}\tilde{\mathbf{M}}_n(\tau)&=\sum_{m\geqslant n}\bar{c}_m\epsilon^{2m}\exp(\mathcal{A}m^{4/3}\tau)\\&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}du\,\exp(-u^2)\Phi(2u\sqrt{\mathcal{A}\tau}),\end{aligned} \tag{19}$$

where the function Φ is given by

$$\Phi(y)=\sum_{m\geqslant n}\bar{c}_m\epsilon^{2m}\exp(m^{2/3}y). \tag{20}$$

If the coefficients $\bar{c}_m$ do not grow too fast (which we shall assume hereafter), the function $\Phi(y)$ is well defined at least for small values of y. Note that for the resummation of Eq. (19) it was not crucial that the power of m is $4/3$. Such a resummation can be performed for any power $\bar{\alpha}<2$ replacing $4/3$.

A more explicit resummation procedure for the right-hand side (rhs) of Eq. (17) can be developed with the help of fractional derivatives. First, let us expand the exponential function

$$\exp(\mathcal{A}\tau m^{4/3})=\sum_{k=0}^{\infty}\frac{(\mathcal{A}\tau m^{4/3})^k}{k!}. \tag{21}$$

Then, writing $\epsilon^{2m}=e^{m\ln\epsilon^2}\equiv e^{\xi m}$, we obtain

$$\tilde{\mathbf{M}}_n(\tau) = \sum_{k=0}^{\infty} \frac{(\mathcal{A}\tau)^k}{k!} \sum_{m \geqslant n} \bar{c}_m m^{4k/3} e^{\zeta m}. \tag{22}$$

We now introduce the Weyl fractional derivative of order q of a function $f(z)$ by the Weyl integral (see, e.g., [32]):

$$\frac{d^q f(z)}{dz^q} \equiv \frac{1}{\Gamma(-q)} \int_{-\infty}^{z} \frac{f(y)dy}{(z-y)^{1+q}}, \tag{23}$$

where for $q>0$ the integral should be properly regularized [32,33] and $\Gamma(-q)$ is the gamma function. For $f(y)=e^{\nu\zeta}$ it takes the form

$$\frac{d^q e^{\nu\zeta}}{d\zeta^q} = \nu^q e^{\nu\zeta}. \tag{24}$$

Substitution of Eq. (24) into Eq. (22) with $\nu=m$ yields

$$\tilde{\mathbf{M}}_n(\tau) = \sum_{k=0}^{\infty} \frac{(\mathcal{A}\tau)^k}{k!} \frac{d^{4k/3}}{d\zeta^{4k/3}} \sum_{m \geqslant n} \bar{c}_m \epsilon^{2m}. \tag{25}$$

If $\bar{c}_m$ are bounded, as shown in Appendix B, by some $\bar{C}_n$, the sum in Eq. (25) is bounded by $\frac{\bar{C}_n e^{2n}}{1-\epsilon^2}$; hence, $\tilde{\mathbf{M}}_n$ is the fractional derivative of some well-defined function presented in Eq. (23) with its regularization.

Therefore, Eqs. (25) and (19) describe the long-time asymptotics of the moments. Therefore, Eq. (17) is an asymptotic expansion and is a good approximation as long as we sum decreasing terms. The condition is (for bounded $\bar{c}_m$)

$$\epsilon^{2m} \exp(\mathcal{A}m^{4/3}\tau) > \epsilon^{2(m+1)} \exp(\mathcal{A}(m+1)^{4/3}\tau). \tag{26}$$

For large m this inequality is $\ln\frac{1}{\epsilon} > \frac{2}{3}\mathcal{A}\tau m^{1/3}$ with $m \geqslant n$. Consequently, for time of the order

$$\tau < \tau_0^{(n)} \equiv \frac{3}{2} \frac{\ln(1/\epsilon)}{\mathcal{A}n^{1/3}}, \tag{27}$$

the nth moment will be dominated by the leading terms and will grow as in the linear case ($\beta=0$).

C. Growth of the second moments

The second moments are of particular interest for the present work. Their growth for a time that is shorter than $\tau_0^{(1)}$ of Eq. (27) is dominated by the leading term: namely, $\tilde{\mathbf{M}}_2 = \bar{c}_2 \epsilon^2 e^{\lambda_{\max}(1)\tau}$. Consequently, for $\tau < \tau_0^{(1)}$,

$$\langle u^2 \rangle = M_{2,0} = \bar{c}_2 \epsilon^2 e^{\lambda_{\max}(1)\tau}. \tag{28}$$

This result was verified numerically. Using the analogy between the stationary Schrödinger equation (1) and the Fokker-Planck equation (4) we identify the generalized Lyapunov exponent $\lambda_{\max}(1)$ with the growth rate (2) as

$$2\gamma = \lambda_{\max}(1) = \lim_{x\to\infty} \lim_{\epsilon\to 0} \frac{\ln\langle \phi^2(x)/\epsilon^2 \rangle}{x} > 0. \tag{29}$$

For $\tau > \tau_0^{(1)}$, nonlinear saturation effects become relevant and the full nonlinear theory should be used.

IV. SUMMARY

In this work the asymptotic behavior of the generalized Lyapunov exponents of the linear Fokker-Planck equation (3) with $\beta=0$ was found analytically and is given by Eq. (12). The resulting expression for the moments (17) is divergent, but it can be resummed in the form (19) or (25). Therefore, for short time $\tau < \tau_0^{(n)}$, the first term in Eq. (17) provides a good approximation of the moments. In particular, for the second moment this result enabled us to identify the generalized Lyapunov exponent of the linear system with the asymptotic growth rate for the nonlinear one: namely,

$$2\gamma = \lambda_{\max}(1). \tag{30}$$

According to the implementation of the method in the manner of Borland, as outlined in [20], this is the decay rate of the stationary states of the random nonlinear equation (1), showing that it is independent of the nonlinearity β. The analysis of the present paper defines the asymptotic region where the wave function is small, a region from where Eq. (30) is found.

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE GENERALIZED LYAPUNOV EXPONENTS

In this appendix, $\lambda_{\max}(n)$, the maximal eigenvalue of the matrix (11) will be evaluated. Following [28] we call this quantity a *generalized Lyapunov exponent*. By elementary dimensional analysis [28], one finds that the following scaling is suggestive:

$$\lambda_{\max}(n) = D^{1/3} L_n\left(\frac{\omega}{D^{2/3}}\right). \tag{A1}$$

In the long-time limit the moments of order $2n$ grow as $e^{\lambda_{\max}(n)t}$. In [28], Zillmer and Pikovsky studied $\lambda_{\max}(n)$ for different values of n: they give exact expressions for $n=2$ for $n\to 0$ (which corresponds to the usual Lyapunov exponent) and consider limiting cases for large values of the dimensionless parameter $\omega/D^{2/3}$. They also study numerically the behavior of $\lambda_{\max}(n)$ as $n\to\infty$, keeping the dimensionless parameter fixed. They found numerically the scaling law

$$\lambda_{\max}(n) \propto n^{\alpha} \quad \text{with } \alpha \simeq 1.4. \tag{A2}$$

The fact that the scaling exponent α is different from 2 implies deviations from Gaussian behavior and breakdown of monoscaling; the consequences of this breakdown of single parameter scaling for conductance distribution were studied by Schomerus and Titov [29]. We also remark that related

studies were also carried out in the context of the harmonic oscillator with random frequency [30,31].

In this appendix, we study analytically the behavior of the generalized Lyapunov exponents in the limit $n \to \infty$. For the special case of $\omega=0$, we prove the following asymptotic formula:

$$\lambda_{\max}(n) \simeq \frac{3}{4} D^{1/3} (2n)^{4/3}. \tag{A3}$$

We shall then argue that this behavior remains valid for any finite value of ω.

We now outline the proof of the scaling equation (A3) for $\omega=0$ by studying the large-n behavior of the coefficients of the characteristic polynomial $P(X)$ of the matrix A_n. Recalling that if $\lambda_{n,1}, \ldots, \lambda_{n,2n+1}$ are the eigenvalues of A_n we have

$$P(X) = \prod_{i=1}^{2n+1} (X - \lambda_{n,i}) = X^{2n+1} - \left(\sum_{i=1}^{2n+1} \lambda_{n,i} \right) X^{2n} + \left(\sum_{i \neq j} \lambda_{n,i} \lambda_{n,j} \right) X^{2n-1} + \cdots + \prod_{i=1}^{2n+1} \lambda_{n,i}. \tag{A4}$$

We also have $\lambda(n)=\max_l(\mathrm{Re}\lambda_{n,l})$. The coefficients of the characteristic polynomial $P(X)$ are symmetric functions of the eigenvalues $\lambda_{n,i}$. Thanks to the Newton formulas, all these coefficients can be written as linear combinations of traces of powers of A_n. Hence, we have

$$P(X) = X^{2n+1} - \mathrm{Tr}(A_n) X^{2n} + \frac{1}{2} \{ [\mathrm{Tr}(A_n)]^2 - \mathrm{Tr}(A_n^2) \} X^{2n-1} + \cdots . \tag{A5}$$

In principle, we can obtain the eigenvalues from the traces of the $2n+1$ powers of A_n. In practice, since we are interested only in the asymptotic behavior of $\lambda_{\max}(n)$, it will be computed from the traces of high powers. From Eq. (11) we observe that $\mathrm{Tr}(A_n)=0$. In the case $\omega=0$, we also have $\mathrm{Tr}(A_n^2)=0$. More generally, for $\omega=0$, we can show that $\mathrm{Tr}(A_n^k) \neq 0$ only when k is a multiple of 3. Indeed, writing

$$A_n = d + g, \tag{A6}$$

with

$$d = \begin{pmatrix} 0 & 2n & 0 & & \\ 0 & 0 & 2n-1 & & \\ & \cdot & \cdot & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

$$g = D \begin{pmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 2 & 0 & 0 & & & & \\ & 6 & 0 & 0 & & & \\ & & 12 & 0 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 2n(2n-1) & 0 & 0 \end{pmatrix}, \tag{A7}$$

we obtain

$$(d+g)^k = \sum_{k_1+k_2=k} W_{k_1,k_2}, \tag{A8}$$

where W_{k_1,k_2} is a product of k factors with k_1 factors equal to d and k_2 factors equal to g, with $k_1+k_2=k$ (there are 2^k such terms because the matrices d and g do not commute). We remark that d is an upper-diagonal band matrix and its nonvanishing terms are all on the band located at level +1 above the diagonal. Similarly g is a lower-triangular band matrix and its nonvanishing terms are all on the band located 2 levels below the diagonal. Therefore a product of k_1 matrices d and k_2 matrices g will have a nonzero diagonal term only if $k_1=2k_2$—i.e., if $k=3k_2$; hence, k is a multiple of 3. For example, we have

$$\begin{aligned} \mathrm{Tr}(A_n^3) &= \mathrm{Tr}(ddg + dgd + gdd) \\ &= 3\mathrm{Tr}(d^2 g) \\ &= 3D \sum_{l=1}^{2n-1} l(l+1)(2n-l+1)(2n-l) \\ &\simeq 3D(2n)^5 \int_0^1 x^2 (1-x)^2 dx \\ &= D \frac{16n^5}{5}. \end{aligned} \tag{A9}$$

More generally, the terms that contribute to $\mathrm{Tr}(A_n^{3k})$ are obtained by taking the product of $2k$ factors d and k factors g written in all possible orders [there are $\frac{(3k)!}{(2k)!k!}$ such terms]:

$$\mathrm{Tr}(A_n^{3k}) = \sum \mathrm{Tr}(W_{2k,k}) = d^{2k} g^k + (\text{permuted terms}). \tag{A10}$$

In particular, we have

$$\begin{aligned} \mathrm{Tr}(d^{2k} g^k) &= D^k \sum_l l(l+1)(l+2) \ldots (l+2k-1) \\ &\quad \times (2n-l+1)(2n-l) \cdots (2n-l-2k+2) \\ &\simeq D^k (2n)^{4k+1} \int_0^1 x^{2k} (1-x)^{2k} dx \\ &= D^k (2n)^{4k+1} \frac{(2k)!(2k)!}{(4k+1)!}. \end{aligned} \tag{A11}$$

The trace of any term $W_{2k,k}$ is given by the same expression at leading order: indeed, the elements of the matrix dg are of order n^3, whereas those of the commutator $[d,g]$ are of order

TABLE I. Behavior of the dominant eigenvalue in the case $\omega=0$, $D=1$.

$2n$	$\lambda_{\max}(n)$	$\frac{3D(2n)^{4/3}}{4\lambda_{\max}(n)}$
10	15	1.077
40	101	1.015
80	257	1.006
100	347	1.003
120	443	1.0021
140	544	1.0022
160	650	1.0018
180	761	1.0016
200	876	1.0013
300	1504.7	1.0010

n^2. The reason is that both dg and gd are triangular with all nonvanishing matrix elements one level below the diagonal of the form $(dg)_{l,l-1}$ and $(gd)_{l,l-1}$. In the center of the matrix, $l \approx n$, a generic term is of the order of $n^3+O(n^2)$. Therefore both dg and gd are dominated by n^3, while $[d,g]$, given by the difference of two such terms, is dominated by the $O(n^2)$ corrections. We know that any $W_{2k,k}$ differs from $d^{2k}g^k$ by a finite number of commutators. Therefore, $\mathrm{Tr}(W_{2k,k}) = \mathrm{Tr}(d^{2k}g^k) + (\text{subleading terms})$. We thus have

$$\mathrm{Tr}(A_n^{3k}) \simeq \frac{D^k(2n)^{4k+1}}{(4k+1)} \frac{(3k)!(2k)!}{(4k)!k!}. \tag{A12}$$

Rewriting this trace in terms of the eigenvalues of the matrix A_n, we obtain

$$\sum_{i=1}^{2n+1} \lambda_{n,i}^{3k} \simeq \frac{D^k(2n)^{4k+1}}{(4k+1)} \frac{(3k)!(2k)!}{(4k)!k!}. \tag{A13}$$

Besides, from Eq. (A12), we deduce that

TABLE II. Behavior of the dominant eigenvalue in the case $\omega=1$, $D=1$.

$2n$	$\lambda_{\max}(n)$	$\frac{3D(2n)^{4/3}}{4\lambda_{\max}(n)}$
10	13.2	1.223
40	95.9	1.072
80	248	1.043
100	336	1.035
120	431	1.031
140	530	1.028
160	635	1.026
180	745	1.023

TABLE III. Behavior of the dominant eigenvalue in the case $\omega=-1$, $D=1$.

$2n$	$\lambda_{\max}(n)$	$\frac{3D(2n)^{4/3}}{4\lambda_{\max}(n)}$
10	18	0.900
40	107.5	0.952
80	267	0.968
100	358	0.972
120	455	0.975
140	557.5	0.977
160	666	0.980
180	777	0.981

$$\frac{[\mathrm{Tr}(A_n^3)]^k}{\mathrm{Tr}(A_n^{3k})} \sim n^{k-1}. \tag{A14}$$

This equation shows that the trace is not dominated by the largest eigenvalue alone: otherwise, this "participation" ratio would be of order 1. Rather, a finite fraction ρ of the eigenvalues has a scaling behavior similar to that of $\lambda_{\max}(n)$ and we can write $\mathrm{Tr}(A_n^{3k}) \sim \rho n \lambda_{\max}(n)^{3k}$. Thus, we deduce from Eq. (A13) the behavior

$$\rho n \lambda_{\max}(n)^{3k} \simeq \frac{D^k(2n)^{4k+1}}{(4k+1)} \frac{(3k)!(2k)!}{(4k)!k!}. \tag{A15}$$

Finally, using the Stirling formula, we obtain for large k, assuming ρn does not vary strongly with n,

$$\lambda_{\max}(n) \simeq D^{1/3}(2n)^{4/3} \left(\frac{(3k)!(2k)!}{(4k)!k!} \right)^{1/3k} \simeq \frac{3}{4} D^{1/3}(2n)^{4/3}. \tag{A16}$$

This ends the proof of Eq. (A3) or (12) in the case $\omega=0$. In Table I the highest eigenvalue of the matrix A_n is computed numerically for various values of $2n$ up to $2n=300$ (for $\omega=0$ and $D=1$). The asymptotic scaling given in Eq. (A3) is well satisfied.

When ω is different from 0, we can still write A_n as a sum of two matrices as in Eq. (A6): the upper-diagonal matrix d remains the same as in Eq. (A7), but the lower-triangular part g' is now given by the previous g plus a band-diagonal matrix $g_2(\omega)$ (with a band-diagonal at level -1), which contains terms proportional to ω. However, in the large-n limit (and keeping the value of ω fixed) the matrix elements of g are much larger than those of $g_2(\omega)$: hence $g' \simeq g$ up to subdominant contributions and the large-n scaling of the maximal eigenvalue is insensitive to ω at leading order; therefore, Eq. (A3), derived for $\omega=0$, remains true for finite values of ω. In Tables II and III we give numerical results $\omega=1$ and $\omega=-1$ (taking $D=1$ for both cases). The scaling behavior, proportional to $(2n)^{4/3}$, is well satisfied and it seems also that the prefactor $3/4$ remains correct.

APPENDIX B: BOUNDNESS OF THE $\bar{c}_n$

Let us assume an expansion [of the initial moments (16)]

$$M_{n.l} = \sum_{m=0}^{\infty} b_{n,l}^{(n+m)} \epsilon^{2(n+m)} \tag{B1}$$

for the lth moment of order n as defined after the matrix (11). The b's are bound (by construction). Let us expand the $c_{n,l}$ of Eq. (16) in powers of ϵ as

$$c_{n,l} = \sum_{m'} \bar{c}_{n,l}^{(m')} \epsilon^{2m'}. \tag{B2}$$

Since only even moments are considered, the powers of ϵ are even. Now we write Eq. (16) in the form

$$M_{n.l} = \sum_{n',l'} c_{n',l'} U_{n',l'}^{(n,l)}, \tag{B3}$$

where we equate the components of the vectors. Now we use the property (14) of the eigenvectors $U_{n',l'}^{(n,l)}$—namely, $U_{n',l'}^{(n,l)}=0$ for $n>n'$—and solve the equation order by order in ϵ.

Take first $n=1$ and leading order in ϵ,

$$b_{1,l}^{(1)} = \sum_{l'} \bar{c}_{1,l'}^{(1)} U_{1,l'}^{(1,l)}, \quad l=0,1,2. \tag{B4}$$

These are three linear equations for the $\bar{c}_{1,l'}$, since the eigenvectors $U_{1,l'}^{(1,l)}$ are given.

Next take $n=2$,

$$b_{1,l}^{(2)} = \sum_{l'=0}^{2} \bar{c}_{1,l'}^{(2)} U_{1,l'}^{(1,l)} + \sum_{l'=0}^{4} \bar{c}_{2,l'}^{(2)} U_{2,l'}^{(1,l)}, \tag{B5}$$

$$b_{2,l}^{(2)} = \sum_{l'=0}^{2} \bar{c}_{1,l'}^{(2)} U_{1,l'}^{(2,l)} + \sum_{l'=0}^{4} \bar{c}_{2,l'}^{(2)} U_{2,l'}^{(2,l)}. \tag{B6}$$

These are eight equations for the $\bar{c}_{2,l'}^{(2)}, \bar{c}_{1,l'}^{(2)}$.

This process can be continued to any order in ϵ. The independence of the eigenvectors implies that finite solutions for the $\bar{c}_{n,l}^{(m')}$ can be obtained by Kramer's rule. Note that $b_{n,l}^{(m')}=0$ for $n>m'$; therefore, also $\bar{c}_{n,l}^{(m')}=0$ for $n>m'$. Therefore,

$$c_{n,l} \sim \bar{c}_n \epsilon^{2n} \tag{B7}$$

for small ϵ.

This shows that the $\bar{c}_n$ are bounded for any finite n, but it does not imply the existence of a uniform bound for all n and m.

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